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On the translations of quasimonotone maps and monotonicity

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Abstract

We show that given a convex subset K of a topological vector space X and a multivalued map $T: K \rightrightarrows X^*$, if there exists a nonempty subset S of X^* with the surjective property on K and $T + w$ is quasimonotone for each $w \in S$, then T is monotone. Our result is a new version of the result obtained by N. Hadjisavvas (Appl. Math. Lett. 19:913-915, 2006).

Keywords: monotone map; pseudomonotone map; quasimonotone map; surjective property

1 Introduction and some definitions

Throughout the paper, X and X^* denote a real topological vector space and the dual space of X, respectively. Suppose $K \subseteq X$ is a nonempty subset of X and $T : K \rightrightarrows X^*$ is a multivalued map from K to X^*. Recall that T is said to be monotone if for all $x^* \in T(x)$, $y^* \in T(y)$ one has

$$\langle x^* - y^*, x - y\rangle \geq 0.$$

T is said to be pseudomonotone and quasimonotone, in the sense of Karamardian (see [1, 2]), respectively, if for any $x^* \in T(x)$, $y^* \in T(y)$ the following implications hold:

$$\langle y^*, x - y\rangle \geq 0 \quad \Rightarrow \quad \langle x^*, x - y\rangle \geq 0$$

and

$$\langle y^*, x - y\rangle > 0 \quad \Rightarrow \quad \langle x^*, x - y\rangle \geq 0.$$

It is clear that a monotone map is pseudomonotone, while a pseudomonotone map is quasimonotone. The converse is not true. If T is pseudomonotone (quasimonotone) and $w \in X^* \setminus \{0\}$, then $T + w$ is not pseudomonotone (quasimonotone) in general. In the case of a single-valued linear map T defined on the whole space $\mathbb{R}^n$, it is known that if $T + w$ is quasimonotone, then T is monotone [2]. Many authors (see, *e.g.*, [4, 5]) extended this result for a nonlinear Gateaux differentiable map defined on a convex subset K (of a Hilbert space) with a nonempty interior.

Recently, Hadjisavvas [3] extended the above result to the multivalued maps defined on a convex subset of a real topological vector space with no assumption of differentiability or even continuity on the map T whose domain need not have a nonempty interior. In this paper, we first introduce the surjective property of a subset of X^* on a segment of K. By using this concept, we can extend the corresponding result obtained in [3]. Before stating the main result, we recall some definitions.

Definition 1 Let x, y be two elements of K. We say that $S \subseteq X^*$ has the surjective property on x and y whenever the following equality holds:

$$\langle S, x-y\rangle = \{\langle x^*, x-y\rangle : x^* \in S\} = \mathbb{R}.$$

Remark that we can consider $x-y$ as a linear functional (denoted by $\widehat{x-y}$) on X^* which is defined by

$$\langle \widehat{x-y}, f\rangle = \langle f, x-y\rangle.$$

Hence if S has the surjective property on x, y, then the image of S under the linear functional $\widehat{x-y}$ is all of the real numbers, and that is why we used the phrase surjective property.

Definition 2 Let $K \subseteq X$ be a nonempty set and $S \subseteq X^*$. We say that S has the surjective property on K if for every $x \in K$ there exists $y \in K$ such that S has the surjective property on x and y.

Definition 3 [3] Let K be a convex subset of X. An element v of X^* is called perpendicular to K if v is constant on K, *i.e.*,

$$\langle v, x\rangle = \langle v, y\rangle, \quad \forall x, y \in K.$$

Also the straight line $S = \{u + tv : t \in \mathbb{R}\}$, where $u, v \in X^*$ with $v \neq 0$, is said to be perpendicular to K if v is perpendicular to K.

Remark 1 If $K \subseteq X$ is a nonempty convex set and $u, v \in X^*$ with v is not perpendicular to K, then the straight line $S = \{u + tv : t \in \mathbb{R}\}$ has the surjective property on K. Indeed, let $x \in K$ be an arbitrary member of K. Because v is not perpendicular to K, there exists $y \in K$ such that $c = \langle v, x-y\rangle \neq 0$. For each $a \in \mathbb{R}$, we put $t = \frac{a-\langle u,x-y\rangle}{c}$ and so $a = \langle u + tv, x-y\rangle$. Hence $\langle S, x-y\rangle = \mathbb{R}$. This means that S has the surjective property. Therefore, v being not perpendicular to K implies the surjective property while the simple example $X = \mathbb{R}^2$, $S = \{(t,t) = (0,0) + (1,1)t : t \in \mathbb{R}\}$ and $K = \{(x,-x) : x \in \mathbb{R}\}$ shows that the converse does not hold in general. In this example, one can see that S has the surjective property and $v = (1,1)$ is perpendicular to K (note $\langle v = (1,1), (x,-x)\rangle = \langle v = (1,1), (y,-y)\rangle = 0$). The notion v is not perpendicular to K, which plays a crucial rule in proving the main results in [3]; while in this note, the surjective property has an essential rule in the main result. Hence one can consider this paper as an improvement of [3] (slightly, of course).

We need the following lemma in the sequel.

Lemma 1 *Let X be a real topological vector space, K a nonempty convex subset of X and $T : K \rightrightarrows X^*$ a multivalued map. Suppose $x, y \in K$, $S \subseteq X^*$ has the surjective property on x, y and $T + w$ is quasimonotone on the line segment $[x, y] = \{tx + (1 - t)y : t \in [0, 1]\}$ for all $w \in S$. Then T is monotone on $[x, y]$.*

Proof We can define an order on $[x, y]$ as follows:

$$a \preceq b \quad \Leftrightarrow \quad t_1 \leq t_2, \quad \text{where } a = x + t_1(y - x), b = x + t_2(y - x).$$

On the contrary, assume T is not monotone on $[x, y]$. So there exist $a, b \in [x, y]$ and $a^* \in T(a)$, $b^* \in T(b)$ with $a \prec b$ and $\langle a^* - b^*, a - b\rangle < 0$. Hence we have

$$\langle a^*, y - x\rangle > \langle b^*, y - x\rangle.$$

Since S is surjective on x, y, there exists $w \in S$ such that

$$\langle a^*, y - x\rangle > \langle w, x - y\rangle > \langle b^*, y - x\rangle.$$

Therefore,

$$\langle a^* + w, y - x\rangle > 0, \qquad \langle b^* + w, y - x\rangle < 0,$$

which is a contradiction. This completes the proof. □

Now we are ready to present the main result.

Theorem 1 *Let X be a real topological vector space, K a nonempty convex subset of X and $T : X \rightrightarrows X^*$ a multivalued map. Assume $S \subseteq X^*$ is connected and has the surjective property on K. If $T + w$ is quasimonotone for all $w \in S$, then T is monotone on K.*

Proof Let $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$ be arbitrary elements. If S is surjective on x, y then, by Lemma 1, T is monotone on $[x, y]$ and the proof is complete. Assume S does not have the surjective property on x, y. So $S(x - y) \neq \mathbb{R}$. Since S has the surjective property on K, then there exists $z \in K$ such that S is surjective on x, z; and since S is connected, then $\langle S, y - z\rangle$ is a connected subset of the real numbers unbounded from above and below, and so it is equal to the real numbers. This means that S has the surjective property on y, z and also on $\frac{x+y}{2}$, z. Therefore, it follows from Lemma 1 that T is monotone on $[y, z]$ and $[\frac{x+y}{2}, z]$. Similarly, T is monotone on the segments $[x, z_s]$ and $[y, z_s]$, for all $s \in]0, 1[$, where $z_s = sz + (1 - s)\frac{x+y}{2}$. Therefore, for any $z_s^* \in T(z_s)$ and $z^* \in T(z)$, we have

$$\langle z_s^*, z_s - x\rangle \geq \langle x^*, z_s - x\rangle, \tag{1}$$

$$\langle z_s^*, z_s - y\rangle \geq \langle y^*, z_s - y\rangle, \tag{2}$$

$$\langle z^*, z - z_s\rangle \geq \langle z_s^*, z - z_s\rangle. \tag{3}$$

From (1) and (2), we deduce that

$$2s\left\langle z_s^*, z - \frac{x + y}{2}\right\rangle \geq \langle x^*, z_s - x\rangle + \langle y^*, z_s - y\rangle. \tag{4}$$

Now from $z - z_s = (1-s)(z - \frac{x+y}{2})$ and (3), we obtain

$$\left\langle z^*, z - \frac{x+y}{2} \right\rangle \geq \left\langle z_s^*, z - \frac{x+y}{2} \right\rangle. \tag{5}$$

Combining (4) and (5), we have

$$2s\left\langle z^*, z - \frac{x+y}{2} \right\rangle \geq \langle x^*, z_s - x\rangle + \langle y^*, z_s - y\rangle.$$

So if in the previous inequality we tend $s \to 0$, then $z_s \to \frac{x+y}{2}$, and hence we deduce

$$0 \geq \langle x^*, y - x\rangle + \langle y^*, y - x\rangle.$$

This means T is monotone and the proof is now complete. □

Remark 1 shows that Theorem 1 is a new version of Theorem 1 in [3], although our proof is, in fact, completely similar to it.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and all authors read and approved the final manuscript.

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